

FAMILIES OF HYPERKÄHLER MANIFOLDS

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Dedicated to Professor Tetsuji Shioda on the occasion of his sixtieth birthday

ABSTRACT. We show the density of the jumping loci of the Picard number of a hyperkähler manifold under small deformation and provide several applications. In particular, we apply this to reveal the structure of hierarchy among all the narrow Mordell-Weil lattices of Jacobian K3 surfaces, which essentially reduces the classification of the Mordell-Weil lattices of Jacobian K3 surfaces - which are no more finite in number - to the classification of those of maximal rank 18.

§0. INTRODUCTION - BACKGROUND AND RESULTS

In their article “Families of K3 surfaces”, R. Borchards, L. Katzarkov, T. Pantev and N. I. Shepherd-Barron have found the following remarkable phenomenon concerning the behaviour of the Picard number under global deformation:

Theorem ([BKPS]). *Any complete family of minimal Kähler surfaces of Kodaira dimension 0 and constant Picard number is isotrivial.* \square

Their proof is a global argument based on the ampleness of the zero locus of an automorphic form on a relevant moduli space and therefore requires the completeness of the base space in essence.

The purpose of this notes is to generalise their Theorem to a local setting for a hyperkähler manifold (Theorem 1) by a quite different method and provide several applications (Corollaries 2, 3, 6, 8 and Examples 4, 5, 7).

A hyperkähler manifold is by definition a simply connected, compact Kählerian manifold F which admits, up to scalar multiple, a unique non-degenerate holomorphic 2-form ω_F and is one of the building blocks of manifolds with trivial first Chern class by the Bogomolov decomposition Theorem ([Be]). A K3 surface is nothing but a hyperkähler manifold of dimension 2. Due to the works of Bogomolov, Beauville and Fujiki, the following results which are well-known for a K3 surface also hold for a hyperkähler manifold of any dimension:

- (1) the existence of the natural primitive integral non-degenerate symmetric bilinear form $(*, *)$ on $H^2(F, \mathbb{Z})$ which induces on $H^2(F, \mathbb{C}) = H^{1,1}(F) \oplus \mathbb{C}\omega_F \oplus \mathbb{C}\bar{\omega}_F$ the polarised Hodge structure of weight two and satisfies $(c_1(L)^2) > 0$ if L is ample ([Be], [Fu2]);

- (2) the local Torelli Theorem for the period map given by the polarised Hodge structure of $H^2(F, \mathbb{Z})$ defined in (1) ([Be]).

Besides original articles, we also refer the readers to [Hu1, Section 1] as an excellent survey about these basics on a hyperkähler manifold.

Throughout this notes, we work over the complex number field \mathbb{C} . We mainly consider a family of hyperkähler manifolds $f : \mathcal{X} \rightarrow \Delta$ over the unit disk Δ with prescribed “polarisation” Λ_0 . We regard Δ as a germ ($0 \in \Delta$) of deformation of the centre fibre $F := \mathcal{X}_0$ and shrink freely whenever it is more convenient for the statement. Therefore, the following three statements are equivalent to one another:

- (i) f is trivial as a family, that is, isomorphic to the product $F \times \Delta$ over Δ ;
- (ii) f is isotrivial, that is, trivial after a finite base change of Δ ;
- (iii) all the fibres of f are isomorphic.

We denote by $\rho(F)$ the Picard number of F , that is, the rank of the Néron-Severi group $NS(F)$. Then $0 \leq \rho(F) \leq N := b_2(F) - 2$. For the precise definition of Λ_0 -polarised family and for our purpose later, we fix an isometric marking $\tau : R^2 f_* \mathbb{Z}_{\mathcal{X}} \simeq \Lambda \times \Delta$, where $\Lambda = (\Lambda, (*, *))$ is a lattice of signature $(3, N - 1)$, and denote the resulting period map by

$$p : \Delta \rightarrow \mathcal{D} := \{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\} \subset \mathbb{P}(\Lambda \otimes \mathbb{C}) \simeq \mathbb{P}^{N+1}.$$

This map is defined by $p(t) := \tau(\omega_{\mathcal{X}_t})$ and is known to be holomorphic. In the case of K3 surfaces, Λ is the K3 lattice $\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ and we have $N = 20$ and $0 \leq \rho(F) \leq 20$. We call a family Λ_0 -polarised if $\tau^{-1}(\Lambda_0) \subset NS(\mathcal{X}_t)$ for all $t \in \Delta$. We do not require hyperbolicity of Λ_0 in apriori and even allow Λ_0 to be $\{0\}$, the later of which is nothing but the case of no fixed “polarisations”. By the term I -topology, we mean the topology on Δ for which the collection of open sets is $\{\emptyset, \Delta, \Delta - \{\text{countably many points}\}\}$. The I -topology is compatible with shrinking. We call a real valued function $y = a(x)$ on a topological space X strictly upper semi-continuous at $x_0 \in X$ if there exists an open neighborhood U of x_0 such that $a(x_0) > a(x')$ for all $x' \in U - \{x_0\}$.

Our main Theorem is as follows:

Theorem 1. *For any non-trivial, Λ_0 -polarised family $f : \mathcal{X} \rightarrow \Delta$ of hyperkähler manifolds, there exists a dense, countable subset $\mathcal{S} \subset \Delta$ in the classical topology such that the function $\rho(t) := \rho(\mathcal{X}_t)$ on Δ is strictly upper semi-continuous at each point of \mathcal{S} in the I -topology but is constant on the complement $\Delta - \mathcal{S}$, provided that all the fibres are algebraic in the case of $\dim F \geq 4$.*

It is an easy fact that \mathcal{S} is at most countable and the essence of Theorem 1 is in the converse: the existence of enough jumping points. Note that by Baire’s category Theorem, $\Delta - \mathcal{S}$ is also dense but uncountable and is “much bigger” than \mathcal{S} . Note also that the same problem does not make much sense for Calabi-Yau manifolds: Indeed, we have $\text{Pic}(X) = H^2(X, \mathbb{Z})$ for them.

Our proof is based on the materials (1) and (2) above and is quite primitive. For this reason, most part of our proof goes through regardless of the dimension of fibres. The only difference at the present between the cases of dimension 2 and higher is a regrettable lack of the criterion for algebraicity in higher dimension: Whether or not the hyperbolicity of $NS(F)$ implies the algebraicity of F . If this is affirmative, then the assumption on algebraicity made at the end of Theorem 1

is not needed. Refer to [Hu2] for positive directions. We shall prove Theorem 1 in Section 1.

Thanks to the existence of the coarse moduli scheme for polarised manifolds due to Viehweg ([Vi]), Theorem 1 deduces the following slightly stronger isotriviality in an algebraic setting:

Corollary 2. *Any smooth family $g : \mathcal{Y} \rightarrow \mathcal{B}$ projective over a normal noetherian irreducible base scheme \mathcal{B} of hyperkähler manifolds with constant Picard number becomes trivial after appropriate étale Galois covering $\mathcal{B}' \rightarrow \mathcal{B}$ of the base scheme. In particular, if in addition $\pi_1^{\text{alg}}(\mathcal{B}) = \{1\}$, then g itself is trivial.*

Here we do not require the completeness of \mathcal{B} .

It is known that there exists a non-isotrivial, smooth projective family of super-singular K3 surfaces over \mathbb{P}^1 . Therefore the corresponding statement in positive characteristic is false even if we assume the base space to be complete. (Refer to [GK] for relevant phenomena in positive characteristics.) We shall prove Corollary 2 in Section 2.

Besides geometrical applications, Theorem 1 also deduces the following lovely result on arithmetic. We shall prove this in Section 2:

Corollary 3. *Let \mathbb{H} be the upper half plane and $GL^+(2, \mathbb{Q})$ the index two subgroup of $GL(2, \mathbb{Q})$ consisting of elements M such that $\det M > 0$. Let $w = \varphi(z)$ be a holomorphic function defined over a neighbourhood of $\tau \in \mathbb{H}$. Assume that $\varphi(\tau) \in \mathbb{H}$. Then, there exists a sequence $\{\tau_k\}_{k=0}^\infty \subset \mathbb{H} - \{\tau\}$ such that $\lim_{k \rightarrow \infty} \tau_k = \tau$ and that τ_k and $\varphi(\tau_k)$ are congruent for each k with respect to the standard, linear fractional action of $GL^+(2, \mathbb{Q})$ on \mathbb{H} .*

In the rest of this Introduction, we shall restrict ourselves to a family of K3 surfaces and see what happens more closely. Explicit construction and proof will be given in Section 3 and Section 2 respectively.

It is reasonable to measure how algebraic a given K3 surface is by the lexicographic order of pairs $(a(S), \rho(S))$. Here $a(S)$ is the algebraic dimension. From this view point, the next example might be of some interest:

Example 4. *There is a family of K3 surfaces $f : \mathcal{X} \rightarrow \Delta$ for which there exists a dense, countable subset $\mathcal{S} \subset \Delta$ such that $a(\mathcal{X}_s) = 2$ and $\rho(\mathcal{X}_s) = 20$, the maximum, for $s \in \mathcal{S}$, but $a(\mathcal{X}_t) = 0$, the minimum, and $\rho(\mathcal{X}_t) = 19$ for $t \notin \mathcal{S}$.*

The values of ρ do not always sweep out some range of integers “continuously”:

Example 5. *There is a family of algebraic K3 surfaces $f : \mathcal{X} \rightarrow \Delta$ for which there exists a dense countable subset \mathcal{S} such that $\rho(\mathcal{X}_s) = 20$ for $s \in \mathcal{S}$, but $\rho(\mathcal{X}_t) = 18$ for $t \notin \mathcal{S}$. In particular, there lacks the intermediate integer 19.*

A Jacobian K3 surface is a K3 surface equipped with an elliptic fibre space structure $\varphi : S \rightarrow \mathbb{P}^1$ with section O . By a local family of Jacobian K3 surfaces, we mean a commutative diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\varphi} & \mathcal{W} & \xrightarrow{\mathcal{O}} & \mathcal{X} \\ f \downarrow & & \downarrow \pi & & \downarrow f \\ & & id & & id \end{array}$$

such that $\varphi_t : \mathcal{X}_t \rightarrow \mathcal{W}_t$ is a Jacobian K3 surface with section \mathcal{O}_t .

The Mordell-Weil lattice $MW(\varphi)$ of φ is the Mordell-Weil group, that is, the group of sections of φ , equipped with Shioda's positive definite, symmetric bilinear form $\langle *, * \rangle$ [Sh2]. This lattice structure on $MW(\varphi)$ made the study of Mordell-Weil groups extremally rich [Sh3]. We refer the readers to [Sh2] and [Sh3] for Mordell-Weil lattices. We denote by $r(\varphi)$ the rank of $MW(\varphi)$. Note that $0 \leq r(\varphi) \leq 18$ for a Jacobian K3 surface [Sh1].

A similar but slightly different jumpinig phenomenon is found for $r(\varphi)$:

Corollary 6. *For any family $f = \pi \circ \varphi : \mathcal{X} \rightarrow \mathcal{W} \rightarrow \Delta$ of Jacobian K3 surfaces, there exists a dense countable subset $\mathcal{S}' \subset \Delta$ in the classical topology such that the function $r(t) := r(\mathcal{X}_t)$ is strictly upper semi-continous at each $s \in \mathcal{S}'$ in the I-topology unless $f : \mathcal{X} \rightarrow \Delta$ is trivial.*

Example 7. *There exists a family $f = \pi \circ \varphi : \mathcal{X} \rightarrow \mathcal{W} \rightarrow \Delta$ of Jacobian K3 surfaces such that $\rho(0) = 20$ but $r(0) = 0$, and that $\rho(t) < 20$ but $r(t) > 0$ for general t in the sense of the I-topology. In particular, $\rho(t)$ is strictly upper semi-continous at $t = 0$ but $r(t)$ is strictly lower semi-continuous at $t = 0$ in the I-topology.*

Example 7 also shows that there is a case where the behaviour of $r(t)$ is not honestly accompanied with that of $\rho(t)$. For comparison, we also note that $\rho(t)$ and $r(t)$ for a family of rational Jacobian surfaces are constant and lower semi-continuous respectively by the stability Theorem.

Our final aim of this notes is to apply Theorem 1 to study the structure of the Mordell-Weil lattices of Jacobian K3 surfaces. By the narrow Mordell-Weil lattice $MW^0(\varphi)$ we mean the sublattice of $MW(\varphi)$ of finite index consisting of the sections which pass through the identity component of each fibre [Sh2]. Contrary to the case of rational Jacobian surfaces, the isomorphism classes of both $MW(\varphi)$ and $MW^0(\varphi)$ for Jacobian K3 surfaces are no more finite ([OS], [Ni]) and the whole pictures of them does not seem so clear even now. Our interest here is to clarify certain relationships among all of $MW(\varphi)$ through Theorem 1:

Corollary 8. *For any given Jacobian K3 surface $\varphi : J \rightarrow \mathbb{P}^1$ of rank $r := r(\varphi)$, there exists a sequence $\{\varphi_m : J_m \rightarrow \mathbb{P}^1\}_{m=r}^{18}$ of Jacobian K3 surfaces such that*

- (1) $\varphi_r : J_r \rightarrow \mathbb{P}^1$ is the original $\varphi : J \rightarrow \mathbb{P}^1$;
- (2) $r(\varphi_m) = m$ for each m ; and
- (3) *there exists a sequence of isometric embeddings:*

$$MW^0(\varphi)(= MW^0(\varphi_r)) \subset MW^0(\varphi_{r+1}) \subset \cdots \subset MW^0(\varphi_{17}) \subset MW^0(\varphi_{18}).$$

In particular, the narrow Mordell-Weil lattice of any Jacobian K3 surface is embedded into the Mordell-Weil lattice of some Jacobian K3 surface of rank 18. Conversely, for every sublattice M of the narrow Mordell-Weil lattice of a Jacobian K3 surface of rank 18, there exists a Jacobian K3 surface whose narrow Mordell-Weil lattice contains M as a sublattice of finite index. Moreover, for each given M there are at most finitely many isomorphism classes of the Mordell-Weil lattices of Jacobian K3 surfaces which contains M as a sublattice of finite index.

This Corollary coarsely reduces the study of $MW(\varphi)$ to those of the maximal rank 18 and is in some sense an analogue of the fact that $MW^0(\varphi)$ of rational Jacobian surfaces are embedded into E_8 . Concerning rational Jacobian surfaces, see [OS].

[OS] together with a correction at the end of Section 3 for the classification, and refer to [Sh3, Part II] for beautiful aspects behind the hierarchy governed by the lattice E_8 . It might be also worthwhile noticing here that a Jacobian K3 surface with Mordell-Weil rank 18 is necessarily “singular” in the sense of Shioda [SI] and that Nishiyama [Ni] has already constructed an infinite series of examples of such Jacobian K3 surfaces. We shall prove Corollary 8 in Section 2.

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§1. PROOF OF THEOREM 1

We employ the same notation and convention as in Introduction.

Lemma (1.1). *There exists a unique primitive sublattice $\Lambda_1 \subset \Lambda$ such that:*

- (1) $\Lambda_0 \subset \Lambda_1 \subset \tau(NS(\mathcal{X}_t))$ for all $t \in \Delta$, and that;
- (2) if $\Lambda \subset \tau(NS(\mathcal{X}_t))$ for all $t \in \Delta$, then $\Lambda \subset \Lambda_1$.

Moreover, there exists a unique, non-empty I -open subset \mathcal{U} such that $\tau(NS(\mathcal{X}_t)) = \Lambda_1$ for each $t \in \mathcal{U}$ and that $\tau(NS(\mathcal{X}_t))$ is strictly bigger than Λ_1 if $t \notin \mathcal{U}$. In particular, the jumping points of ρ is at most countable.

Proof. Consider all the primitive sublattices Λ_n ($n \in \mathcal{N}$) of Λ which contain Λ_0 and set $\Delta(n) := \{t \in \Delta \mid \tau(NS(\mathcal{X}_t)) = \Lambda_n\}$. Then $\Delta = \sqcup_{n \in \mathcal{N}} \Delta(n)$, because $NS(\mathcal{X}_t)$ is primitive in $H^2(\mathcal{X}_t, \mathbb{Z})$. Recall that \mathcal{N} is countable but Δ is uncountable and that countable union of countable sets is again countable. Therefore there exists

an element of \mathcal{N} , which we denote by 1, such that $\Delta(1)$ is uncountable. By the Lefschetz (1,1)-Theorem, $p(t) \in \Lambda_1^\perp \otimes \mathbb{C}$ for all $t \in \Delta(1)$. Then $p(\Delta)$ must be contained in the linear space defined by $(\Lambda_1.*) = 0$ in $\mathbb{P}(\Lambda \otimes \mathbb{C})$, because p is holomorphic. Therefore, again, by the Lefschetz (1,1)-Theorem, we have $\Lambda_1 \subset \tau(NS(\mathcal{X}_t))$ for all $t \in \Delta$. Set $\mathcal{U} := \{t \in \Delta \mid \tau(NS(\mathcal{X}_t)) = \Lambda_1\}$. Then, by the choice of Λ_1 , we know that $\mathcal{U} \neq \emptyset$. In addition, since $\Lambda_1 \subset \tau(NS(\mathcal{X}_t))$ for all $t \in \Delta$ and since Λ_1 is primitive, we see that $\tau(NS(\mathcal{X}_t)) \neq \Lambda_1$ if and only if there exists an element $v \in \Lambda - \Lambda_1$ such that $p(t) \in v^\perp$, that is, $(p(t).v) = 0$. Set $\mathcal{S} := \cup_{v \in \Lambda - \Lambda_1} (p(\Delta) \cap v^\perp)$. Then $\mathcal{U} = \Delta - p^{-1}(\mathcal{S})$. Since $\mathcal{U} \neq \emptyset$, we have $p(\Delta) \neq p(\Delta) \cap v^\perp$ for each v . Since p is holomorphic, this implies that $p(\Delta) \cap v^\perp$ is at most countable. Hence \mathcal{S} is also countable. Therefore, \mathcal{U} is I -open, again by the fact that p is holomorphic. Now we are done. \square

As in Introduction, we set $N := b_2(F) - 2$, where F is the centre fibre of f . By abuse of notation, we denote the rank of a lattice L by $\dim L$.

Lemma (1.2). *Let Λ_1 be the lattice found in (1.1). Then $\dim \Lambda_1 \leq N - 1$. In other words, f is trivial if $\dim \Lambda_1 = N$.*

Proof. We have $\dim \Lambda_1 \leq N$. Assume for a contradiction that $\dim \Lambda_1 = N$. Then $\Lambda_1 \otimes \mathbb{R} \simeq H^{1,1}(F, \mathbb{R})$ and $\Lambda_1^\perp \otimes \mathbb{R} \simeq \mathbb{R}\langle \text{Re}(\omega_F), \text{Im}(\omega_F) \rangle$ by the dimension reason. In particular, Λ_1^\perp is positive definite and is of rank 2. In addition, since $(p(t).\Lambda_1) = 0$, we have $p(\Delta) \subset \{[\omega] \in \mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C}) \mid (\omega.\omega) = 0, (\omega.\bar{\omega}) > 0\} \subset \mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C}) \simeq \mathbb{P}^1$, where we regard $\mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C})$ as the linear subspace of $\mathbb{P}(\Lambda \otimes \mathbb{C})$ defined by $(\Lambda_1.*) = 0$. Note that the equation $(\omega.\omega) = 0$ has at most two solutions in \mathbb{P}^1 , because $(*,*)$ is positive definite on Λ_1^\perp and therefore is non-degenerate. Since Δ is connected and p is continuous, this implies that $p(\Delta)$ consists of one point. That is, the period map p is constant. Hence, by the local Torelli Theorem, the fibres of f are all isomorphic and f is then trivial around 0. However this contradicts our assumption. \square

Set $\mathcal{S} := \Delta - \mathcal{U}$, where \mathcal{U} is same as in (1.1). The next Lemma completes the proof:

Lemma (1.3). *\mathcal{S} is dense in Δ in the classical topology.*

Convention for argument of (1.3) and (1.4). In order to prove (P) by the argument by contradiction, we first assume that (P) does not hold and proceed our argument until we reach a contradiction. However, we often meet a situation where we need to prove another property (Q) again by the argument by contradiction. Assume (Q) does not hold, then \dots . In this case, if we reach a contradiction, then the logical conclusion at this stage should be written: Either (P) or (Q) does hold. However, if (P) does hold but (Q) does not hold, then this itself gives a contradiction to the logic we have proceeded, because this time “(Q) does not hold” is the truth, and we have nothing to do more! For this reason, and especially for simplicity of description, our proof below is written as if the conclusion at that stage would be that: the *last* property (Q) does hold. Please accept this logically harmless convention in the proof below.

Proof. Assume to the contrary that \mathcal{S} is not dense. Then, $\overline{\mathcal{S}} \neq \Delta$, where the closure is taken inside Δ . Since $\overline{\mathcal{S}} \subset \Delta$, there then exists a point $P \in \Delta - \overline{\mathcal{S}}$. Since $\Delta - \overline{\mathcal{S}}$ is open in Δ , by the definition of the induced topology, there exists an open subset U of \mathbb{C} such that $U \cap \Delta = \Delta - \overline{\mathcal{S}}$. Since Δ is open in \mathbb{C} , we see

that $U \cap \Delta$ is also open in \mathbb{C} . Therefore, there exists a small disk Δ_P centered at P such that $P \in \Delta_P \subset \Delta - \overline{\mathcal{S}}$. Then, by the definition of \mathcal{S} and by the inclusions $\Delta_P \subset \Delta - \overline{\mathcal{S}} \subset \Delta - \mathcal{S} = \mathcal{U}$, we see that $\tau(NS(\mathcal{X}_t)) = \Lambda_1$ for all $t \in \Delta_P$ by (1.1).

Claim (1.4). $p|_{\Delta_P}$ is constant.

Once we get (1.4), then the result (1.3) follows. Indeed, constantness of p on Δ_P implies the constantness of p on the whole Δ , because p is holomorphic. Then, by the local Torelli Theorem, all the fibres are isomorphic. This in particular implies that p is trivial around 0, a contradiction.

Proof of (1.4). Assume to the contrary that $p|_{\Delta_P}$ is not constant. Assume that $\mathcal{X}_{P'}$ of dimension 2 is not algebraic for some $P' \in \Delta_P$. Then there exists $P'' \in \Delta_P$ such that $\mathcal{X}_{P''}$ is algebraic. This is due to [Fu1, Theorem 4.8 (2)] and is valid in any dimension. Then on the one hand, $\Lambda_1 \simeq NS(\mathcal{X}_{P'})$ is negative semi-definite by the algebraicity criterion for surface, but on the other hand $\Lambda_1 \simeq NS(\mathcal{X}_{P''})$ is hyperbolic, a contradiction. Therefore, $\mathcal{X}_{P'}$ is algebraic for all $P' \in \Delta_P$ even in the case of dimension 2. *From now on, dimension of fibre is arbitrary.* Note that a Kähler Moishezon manifold is projective. Therefore, by the algebraicity of fibre, Λ_1 contains an element a such that $(a, a) > 0$. Then by the Schwartz inequality, Λ_1 is non-degenerate and even hyperbolic, because $\Lambda_1 \subset \tau(H^{1,1}(F, \mathbb{R}))$ and $\tau(H^{1,1}(F, \mathbb{R}))$ is of index $(1, *)$. Let us choose a holomorphic coordinate z of Δ_P centered at P . By the Lefschetz (1,1)-Theorem, we have $p|_{\Delta_P} : \Delta_P \rightarrow \{[\omega] \in \mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C}) | (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\} \subset \mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C}) \simeq \mathbb{P}^n$, where we again regard $\mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C})$ as a linear subspace of $\mathbb{P}(\Lambda \otimes \mathbb{C})$ defined by $(\Lambda_1, *) = 0$. Note that $n \geq 2$ by (1.2). Let us fix an integral basis of Λ_1^\perp and write $p|_{\Delta_P}$ as $p(z) = [F_0(z) : F_1(z) : F_2(z) : \cdots : F_n(z)]$. We may assume without loss of generality that $F_0(0) \neq 0$. Then we may rewrite p under the inhomogeneous coordinates $x_i := X_i/X_0$ as $z \mapsto [1 : f_1(z) : f_2(z) : \cdots : f_n(z)]$ around P . In this expression, $f_i(z)$ are holomorphic functions of z . Assume that $f_i(z)$ are all constant. Then p is constant around P and therefore constant on the whole Δ_P . However, this contradicts our assumption that p is not constant on Δ_P . Therefore, some $f_k(z)$ is not constant. Then by changing the order of the coordinates, we may assume without loss of generality that $f_1(z)$ is not constant. Since $\dim_{\mathbb{R}} \mathbb{C} = 2$, the three elements $1, f_1(0), f_k(0)$ in \mathbb{C} are linearly dependent over \mathbb{R} for each $k \in \{2, \dots, n\}$. Therefore, there exists $(r_{0,k}, r_{1,k}, r_{2,k}) \in \mathbb{R}^3 - \{\vec{0}\}$ such that $r_{0,k} \cdot 1 + r_{1,k} f_1(0) + r_{2,k} f_k(0) = 0 - (*)$. In what follows, for $\vec{a} := (a_0, a_1, a_2, \dots, a_n) \in \mathbb{R}^{n+1} - \{\vec{0}\}$, we denote the hyperplane defined by $a_0 X_0 + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n = 0$ in $\mathbb{P}(\Lambda_1^\perp \otimes \mathbb{C})$ by $H(\vec{a})$ and put $f_{\vec{a}}(z) := a_0 + a_1 f_1(z) + a_2 f_2(z) + \cdots + a_n f_n(z)$. Set $\vec{r}_k := (r_{0,k}, r_{1,k}, 0, \dots, 0, r_{2,k}, 0, \dots, 0)$, where $r_{2,k}$ is located at same position as X_k in $[X_0 : X_1 : \cdots : X_n]$. In this terminology, the previous equation (*) is equivalent to $p(0) \in H(\vec{r}_k)$ and is also equivalent to $f_{\vec{r}_k}(0) = 0$. In particular, $p(\Delta_P) \cap H(\vec{r}_k) \neq \emptyset$. Assume that for some k , we have $p(\Delta_P) \not\subset H(\vec{r}_k)$, that is, $f_{\vec{r}_k}(z) \not\equiv 0$. Since 0 is a zero of $f_{\vec{r}_k}(z)$, by using the discreteness of zeros of a non-zero holomorphic function, we may choose a small circle $\gamma \subset \Delta_P$ around $z = 0$ such that $f_{\vec{r}_k}(z)$ has no zeros on γ . Set $K := \min\{|f_{\vec{r}_k}(z)| | z \in \gamma\}$ and $M := \max\{|f_i(z)| | z \in \gamma, i = 0, 1, \dots, n\}$, where we define $f_0(z) \equiv 1$. Note that $K > 0$ and $M > 0$. Then, by using the triangle inequality, we see that $|f_{\vec{r}_k}(z) - f_{\vec{a}}(z)| < |f_{\vec{r}_k}(z)|$ on γ provided that $|\vec{a} - \vec{r}_k| < KM^{-1}(n+1)^{-1}$. Denote by U the open disk such that $\partial U = \gamma$. Then, by Rouché's Theorem, the cardinalities of zeros counted with multiplicities

on U are the same for $f_{\vec{r}_k}$ and $f_{\vec{a}}$. Therefore, $f_{\vec{a}}$ also admits a zero on U , because 0 is a zero of $f_{\vec{r}_k}$. Since $\mathbb{Q}^{n+1} - \{\vec{0}\}$ is dense in $\mathbb{R}^{n+1} - \{\vec{0}\}$ and since $KM^{-1}(n+1)^{-1} > 0$, there then exists $\vec{q} \in \mathbb{Q}^{n+1} - \{\vec{0}\}$ such that $f_{\vec{q}}(z)$ admits a zero on U . Let us denote this zero by $Q \in U(\subset \Delta_P)$. Then, $f_{\vec{q}}(Q) = 0$. Or in other words, $p(Q) \in H(\vec{q})$. Recall that Λ_1^\perp is non-degenerate because so is Λ_1 . Note also that our homogeneous coordinates $[X_0 : X_1 : \cdots : X_n]$ is chosen with respect to an integral basis of Λ and the rational linear equations $(\Lambda_1.*) = 0$. Therefore, there exists a rational vector $0 \neq v \in \Lambda_1^\perp \otimes \mathbb{Q}$ such that $(v.*)$ gives the same linear space as $H(\vec{q})$. Then $(v.p(Q)) = 0$. This together with the Lefschetz (1,1)-Theorem implies that $mv \in \tau(NS(\mathcal{X}_Q))$, where $m > 0$ is a suitable integer. Note also that $mv \notin \Lambda_1$ again because of the non-degeneracy of Λ_1 . Then $\tau(NS(\mathcal{X}_Q))$ becomes strictly bigger than Λ_1 . On the other hand, since $Q \in \Delta_P \subset \Delta - \mathcal{S} = \mathcal{U}$, we have $\tau(NS(\mathcal{X}_Q)) = \Lambda_1$ by (1.1), a contradiction. Therefore $p(\Delta_P)$ is contained in the intersection of the $(n-1)$ -hyperplanes $H(\vec{r}_k)$ given by $r_{0,k}X_0 + r_{1,k}X_1 + r_{2,k}X_k = 0$, $k = 2, 3, \dots, n$. In other words, we have $r_{0,k} + r_{1,k}f_1(z) + r_{2,k}f_k(z) \equiv 0$ for all k . Assume that $r_{2,k} = 0$ for some k . Then $r_{0,k} + r_{1,k}f_1(z) \equiv 0$. Therefore $r_{0,k} = r_{1,k} = 0$ by the non-constantness of $f_1(z)$. However, this contradicts $(r_{0,k}, r_{1,k}, r_{2,k}) \neq (0, 0, 0)$. Therefore $r_{2,k} \neq 0$ for any k . Hence the intersection of these $(n-1)$ -hyperplanes $H(\vec{r}_k)$ is a line $L \simeq \mathbb{P}^1$ by the rank reason. Note that this L is defined over \mathbb{R} in \mathbb{P}^n . Then, we may choose a real basis of $\Lambda_1^\perp \otimes \mathbb{R}$ under which the coordinate description of p is of the form $p(z) = [1 : g(z) : 0 : \cdots : 0]$. Assume that $g(z)$ is constant. Then p is also constant, a contradiction to the assumption that $p(z)$ is constant on Δ_P . Therefore $g(z)$ is not constant. Let us consider the matrix representation of the symmetric bilinear form $(*.*)|_{\Lambda_1^\perp}$ under this real basis and denote it by $(x, y) = x(c_{ij})x^t$. Here (c_{ij}) is a real symmetric matrix. Since $(p(z).p(z)) = 0$, we have

$$c_{00} + 2c_{10}g(z) + c_{11}g(z)^2 \equiv 0 (**).$$

Differentiating (**) by z , we get $(c_{10} + c_{11}g(z))g'(z) \equiv 0$. If $g'(z) \equiv 0$, then $g(z)$ is constant, a contradiction. Hence $c_{10} + c_{11}g(z) \equiv 0$, because $\mathbb{C}\{z\}$ is an integral domain. Since $g(z)$ is not constant, this implies that $c_{11} = c_{10} = 0$. Then, we also get $c_{00} = 0$ by substituting $c_{11} = c_{10} = 0$ to the equation (**). Therefore, we have $(p(z).p(z)) = c_{00} + c_{10}(g(z) + \overline{g(z)}) + c_{11}g(z)\overline{g(z)} = 0 + 0 + 0 = 0$. Since we changed coordinates only over \mathbb{R} , we still have $[\overline{\omega_{\mathcal{X}_z}}] = [\overline{p(z)}] = [1 : \overline{g_1(z)} : 0 : \cdots : 0]$. Therefore, we have $(p(z).p(z)) = |c(z)|^2(\omega_{\mathcal{X}_z}.\overline{\omega_{\mathcal{X}_z}})$, where $c(z)$ is a nowhere vanishing complex function adjusting ambiguities of the choice of ratio. Since $(\omega_{\mathcal{X}_z}.\overline{\omega_{\mathcal{X}_z}}) > 0$ by the definition of the period map, we have then $(p(z).p(z)) > 0$. However, this contradicts the previous equality $(p(z).p(z)) = 0$. Now we are done. \square

Remark (1.5). In general, given a non-constant holomorphic map $g : \Delta \rightarrow \mathbb{P}^n$, the condition $h(\Delta) \cap H(\vec{a}) \neq \emptyset$ is not open around \vec{a} if this point satisfies $h(\Delta) \subset H(\vec{a})$. For example, consider the holomorphic map $h(z) = [1 : -1 : z]$. Then $h(\Delta) \subset H((1, 1, 0))$ but $h(\Delta) \cap H((1 - \epsilon, 1, 0)) = \emptyset$ for any $\epsilon \neq 0$. \square

§2. PROOF OF COROLLARIES

In this section, we shall prove all Corollaries mentioned in Introduction. Here we again employ the same notation and convention as in Introduction.

Proof of Corollary 2. Let us first show that all the fibres of g are isomorphic. Only for this purpose, we may assume \mathcal{B} is smooth because of the existence of resolution due to Hironaka and may work in the complex analytic category. Fix $P \in \mathcal{B}$ and take an arbitrary $Q \in \mathcal{B}$. Then we may join P and Q by a chain of small disks Δ_i , because \mathcal{B} is now assumed to be smooth and is noetherian and connected. By applying the contraposition of Theorem 1 for each $g|g^{-1}(\Delta_i) : g^{-1}(\Delta_i) \rightarrow \Delta_i$ and using the assumption of constantness of Picard numbers, we see that all the fibres of $g|g^{-1}(\Delta_i)$ are isomorphic for each Δ_i . Therefore \mathcal{Y}_Q is isomorphic to \mathcal{Y}_P as a complex manifold. Since both are projective, \mathcal{Y}_Q is then isomorphic to \mathcal{Y}_P as an algebraic variety by Chow's Theorem. *From now on, we stick to the original given base scheme \mathcal{B} and work in the algebraic category.* Let us denote by F the hyperkähler manifold to which all the fibres of g are now known to be isomorphic. Since g is projective, we may choose a relatively very ample divisor \mathcal{H} on \mathcal{Y} . Then, we may consider a morphism $\pi : \mathcal{B} \rightarrow \mathcal{M}$ defined by $b \mapsto [(\mathcal{Y}_b, \mathcal{H}|_{\mathcal{Y}_b})]$, where \mathcal{M} is an irreducible component of the coarse moduli scheme of polarised hyperkähler manifolds. The existence of \mathcal{M} is guaranteed by Viehweg ([Vi]) as a very special case of his general Theory. Since $c_1 : \text{Pic}(F) \rightarrow H^2(F, \mathbb{Z})$ is injective for a hyperkähler manifold, and since $H^2(F, \mathbb{Z})$ is countable as a set, we see that F admits only countably many polarisations. In particular, the isomorphism classes of $(\mathcal{Y}_b, \mathcal{H}|_{\mathcal{Y}_b})$ consists of at most countably many points. Therefore π must be constant, because \mathcal{B} is irreducible and we are working over the uncountable base field \mathbb{C} . Set $m := \pi(\mathcal{B})$. Then m is a single closed point of \mathcal{M} . If \mathcal{M} happens to be a fine moduli space, then g is obtained by the pull back of the universal family $u : \mathcal{U} \rightarrow \mathcal{M}$ by π . Hence, $g : \mathcal{Y} \rightarrow \mathcal{B}$ is the pull back of the single element $\mathcal{U}_m \rightarrow \{m\}$ and therefore is globally trivial. Even in the general case, it is again known by E. Viehweg that we may work as if there were a universal family on \mathcal{M} in the following sense: There exist three objects: a reduced normal scheme \mathcal{M}' ; a finite automorphism group Γ of \mathcal{M}' such that $\mathcal{M}'/\Gamma = \mathcal{N}$, where \mathcal{N} is the normalisation of the reduction $(\mathcal{M})_{\text{red}}$; and the family $u : \mathcal{U} \rightarrow \mathcal{M}'$ which is universal over the original \mathcal{M} . (Refer to [Vi, Section 9, Theorem 9.25] for details. We just notice here that the assumption made in [Vi, Theorem 9.25] is satisfied if \mathcal{M} is a coarse moduli scheme of polarised manifolds of nef canonical classes, which the author learned from E. Viehweg.) Let us apply to our setting. Since \mathcal{B} is normal, π factors through \mathcal{N} . we denote this factorisation by $\pi_{\mathcal{N}} : \mathcal{B} \rightarrow \mathcal{N}$. Since the natural map $\mathcal{N} \rightarrow \mathcal{M}$ is finite and since \mathcal{B} is irreducible, $\pi_{\mathcal{N}}(\mathcal{B})$ is also a single closed point. Let us take an irreducible component \mathcal{B}' of the fibre product $\mathcal{B} \times_{\mathcal{N}} \mathcal{M}'$ which dominates \mathcal{B} . Then, the image of the fibre product map $\mathcal{B}' \rightarrow \mathcal{M}'$ is again a single closed point, because $\mathcal{M}' \rightarrow \mathcal{N} = \mathcal{M}'/\Gamma$ is finite and \mathcal{B}' is irreducible. Let us denote this point by m' . Then the family $u_{\mathcal{B}'} : \mathcal{U}_{\mathcal{B}'} \rightarrow \mathcal{B}'$ obtained from $u : \mathcal{U} \rightarrow \mathcal{M}'$ by the fibre product map $\mathcal{B}' \rightarrow \mathcal{M}'$ is nothing but the pull back of the single $\mathcal{U}_{m'} \rightarrow \{m'\}$. In particular, this is the trivial family: $p_1 : \mathcal{B}' \times F \rightarrow \mathcal{B}'$. Since, $u : \mathcal{U} \rightarrow \mathcal{M}'$ is universal over \mathcal{M} , the other pull back family $\mathcal{B}' \times_{\mathcal{B}} \mathcal{Y} \rightarrow \mathcal{B}'$ must coincide with the previous family and therefore is trivial. This already shows the isotriviality in the usual sense. Indeed, $\mathcal{B}' \rightarrow \mathcal{B}$ is finite and is even Galois, because so is $\mathcal{M}' \rightarrow \mathcal{M}'/\Gamma$. However, for our stronger version, we need to find a trivialisation by *étale* base change, which we will do from now by using the special fact that our family is a hyperkähler family (but not much). Let us denote the Galois group of $\mathcal{B}' \rightarrow \mathcal{B}$ by K . Then, $g : \mathcal{Y} \rightarrow \mathcal{B}$ is the quotient $(p_1 : \mathcal{B}' \times F \rightarrow \mathcal{B}')/K$, where the action of K on $\mathcal{B}' \times F$ is the action induced by the pull back $\mathcal{B}' \rightarrow \mathcal{B}$. Therefore, for each

$k \in K$, the action of k on $\mathcal{B}' \times F$ is of the form $k : (b', f') \mapsto (k_1(b'), k_2(b', f'))$. Then, by fixing k and varying $b' \in \mathcal{B}'$, we obtain a morphism $\mathcal{B}' \rightarrow \text{Aut}(F)$ defined by $b' \mapsto k_2(b', *)$. Since F is hyperkähler, ω_F gives an isomorphism $T_F \simeq \Omega_F^1$. Therefore $H^0(T_F) \simeq H^0(\Omega_F^1) = 0$, where the last equality is because $\pi_1(F) = 0$. Hence $\text{Aut}(F)$ is discrete. Since \mathcal{B}' is irreducible, $\mathcal{B}' \rightarrow \text{Aut}(F)$ is then constant. This means that the action of $k \in K$ is of the form $k : (b', f') \mapsto (k_1(b'), k_2(f'))$. In particular, we may speak of the homomorphism $p_2 : K \rightarrow \text{Aut}(F)$ given by $k = (k_1, k_2) \mapsto k_2$. Assume that there exists a point $b' \in \mathcal{B}'$ such that its stabiliser is non-trivial. Write this stabiliser group by H . Then, H acts on $F = \{b'\} \times F$ and induces an isomorphism $F \simeq F/H$, because $g : \mathcal{Y} \rightarrow \mathcal{B}$ is also a family of F . Since the canonical class of F is trivial, the ramification formula implies that there are no ramification divisors of $F \rightarrow F/H$. Therefore, $F \rightarrow F/H$ is étale by the purity of the branch loci, because both source and target are smooth. Then, the quotient map $F \rightarrow F/H$ is an isomorphism, because $F/H \simeq F$ is simply connected. Hence, $\text{Im}(p_2|_H : H \rightarrow \text{Aut}(F)) = \{id\}$. Therefore, the stabiliser group of each point of \mathcal{B}' is all contained in $M := \text{Ker}(p_2 : K \rightarrow \text{Aut}(F))$. Note that M is a normal subgroup of K . Now, by the definition of M , we see that $g : \mathcal{Y} \rightarrow \mathcal{B}$ is isomorphic to the quotient of $p_1 : (\mathcal{B}')/M \times F \rightarrow (\mathcal{B}')/M$ by K/M . Moreover, by the construction, we see that $p_1 : (\mathcal{B}')/M \times F \rightarrow (\mathcal{B}')/M$ has no point which admits a non-trivial stabiliser in K/M any more. Therefore, the action of K/M on $(\mathcal{B}')/M$ is free. Hence $p_1 : (\mathcal{B}')/M \times F \rightarrow (\mathcal{B}')/M$ provides a desired trivialisation. \square

Proof of Corollary 3. Let us denote by E_w the elliptic curve of period $w \in \mathbb{H}$. Choose a small neighbourhood $\varphi(\tau) \in \Delta_2 \subset \mathbb{H}$ on which we have a family of elliptic curves $g : \mathcal{G} \rightarrow \Delta_2$ such that $\mathcal{G}_w = E_w$ and that g has a section. Since φ is holomorphic, we may also choose a small neighbourhood $\tau \in \Delta \subset \mathbb{H}$ such that $\varphi(\Delta) \subset \Delta_2$ and that there exists a family of elliptic curves $h : \mathcal{H} \rightarrow \Delta$ which satisfies that $\mathcal{H}_z = E_z$ and admits a section. By pulling back $g : \mathcal{G} \rightarrow \Delta_2$ by φ and taking the fibre product, we obtain a family of abelian surfaces $a : \mathcal{A} := \mathcal{H} \times_{\Delta} \varphi^* \mathcal{G} \rightarrow \Delta$. Here we have $\mathcal{A}_z = E_z \times E_{\varphi(z)}$. Since a admits a section, regarding this section as its 0-section, we may consider the inversion ι of \mathcal{A} over Δ . Dividing \mathcal{A} by ι and taking its crepant resolution, we obtain a family of Kummer surfaces $f : \mathcal{X} \rightarrow \Delta$. By construction, we have $\mathcal{X}_z = \text{Km}(E_z \times E_{\varphi(z)})$. This family f is not a trivial family. Indeed, if f is a trivial family, then there exists a K3 surface S such that $S \simeq \text{Km}(E_z \times E_{\varphi(z)})$ for all $z \in \Delta$. Note that the Kummer surface structures on S , that is, the isomorphism classes of abelian surfaces A such that $S \simeq \text{Km}(A)$, are determined by the choices of 16 disjoint smooth rational curves on S and that there are at most countably many smooth rational curves on a K3 surface. Therefore, there exist at most countably many isomorphism classes of such A . Denote all of them by A_i ($i \in \mathbb{N}$) and set $A_i = \mathbb{C}^2/\Lambda_i$. For each A_i , the product structures on A_i , that is, the structures of decompositions $A_i = E_i \times F_i$, are also countably many, because subtori of A_i are determined by the choices of sublattices of Λ_i . In conclusion, there are at most countably many isomorphism classes of pairs (E, F) such that $S \simeq \text{Km}(E \times F)$. However, the set of the isomorphism classes of E_z ($z \in \Delta$) are uncountable, a contradiction. Therefore, our family $f : \mathcal{X} \rightarrow \Delta$ is not trivial. Recall by [SM] that $\rho(\mathcal{X}_z) = 18$ if E_z and $E_{\varphi(z)}$ are not isogenous and that $\rho(\mathcal{X}_z) \geq 19$ if E_z and $E_{\varphi(z)}$ are isogenous. Then, by Theorem 1, there exists a dense subset $\mathcal{S} \subset \Delta$ such that $\rho(\mathcal{X}_s) \geq 19$ for $s \in \mathcal{S}$, that is, E_s and $E_{\varphi(s)}$ are isogenous for $s \in \mathcal{S}$. Therefore, any sequence in $\mathcal{S} \setminus \{\tau\}$ converging to τ satisfies

our requirement. \square

Proof of Corollary 6. Since both assumption and conclusion are compatible with shrinking of Δ , we do this freely whenever it is convenient. Since $\pi : \mathcal{W} \rightarrow \Delta$ is \mathbb{P}^1 -bundle, π is in particular a smooth morphism. Therefore, we may find three mutually disjoint analytic local sections of π around 0. Then $\mathcal{W} = \mathbb{P}^1 \times \Delta$ by the same argument as in the Hartshorne's book. By Theorem 1, there exists a dense countable subset $\mathcal{S} \subset \Delta$ such that $\rho := \rho(\mathcal{X}_t)$ is constant for $t \in \Delta - \mathcal{S}$ but $\rho(\mathcal{X}_s) > \rho$ for $s \in \mathcal{S}$. Let $\mathcal{D} \subset \mathbb{P}^1 \times \Delta$ be the discriminant locus of φ . In order to describe \mathcal{D} , let us consider the Weierstrass model of $\varphi : \mathcal{X} \rightarrow \mathcal{W}$ with respect to \mathcal{O} and write the equation as $y^2 = x^3 + a(w, t)x + b(w, t)$, where w is the inhomogeneous coordinate of \mathbb{P}^1 and t is the coordinate of Δ . Then \mathcal{D} is given by (the reduction of) the equation $4a(w, t)^3 + 27b(w, t)^2 = 0 - (*)$. By construction, both $a(w, t)$ and $b(w, t)$ are polynomials with respect to w . Therefore the restriction map $\pi|_{\mathcal{D}} : \mathcal{D} \rightarrow \Delta; (w, t) \mapsto t$ has at most finitely many such bad points $P \in \mathcal{D}$ that $\pi|_{\mathcal{D}}$ is not smooth at P . Denote by $\mathcal{T} \subset \Delta$ the set of the image of these bad points. This is then a finite set. In addition, since the type of non-multiple singular fibres are uniquely determined by the local monodromy, the singular fibres of the fibrations $\varphi_t : \mathcal{X}_t \rightarrow \mathbb{P}^1$ ($t \in \Delta - \mathcal{T}$) are exactly the same regardless of t . Write them by T_i ($i = 1, \dots, n$) and denote by m_i the number of the irreducible components of T_i . Then, by Shioda's formula [Sh1], we have $r(\varphi_t) = \rho(\mathcal{X}_t) - 2 - \sum_{i=1}^n (m_i - 1)$ for $t \in \Delta - \mathcal{T}$. Set $\mathcal{S}' := \mathcal{S} - \mathcal{T}$. Then, at each point of \mathcal{S}' , the function $r(t)$ is strictly upper semi-continuous in the I -topology. Since \mathcal{S} is countable and dense (in the classical topology) and \mathcal{T} is a finite set, \mathcal{S}' is also countable and dense. Therefore, this \mathcal{S}' provides a desired set. \square

Proof of Corollary 8. First we shall show the existence of a sequence in the statement. We may assume that $r \leq 17$. Let us consider the Kuranishi family $k : (J \subset \mathcal{U}) \rightarrow (0 \in \mathcal{K})$ of J . This is a germ of the universal deformation of J and is known to be smooth of dimension 20. Therefore, \mathcal{K} is realised as an open neighbourhood of $0 \in H^1(J, T_J)$ and is assumed to be a small polydisk in \mathbb{C}^{20} . Then $R^2k_*\mathbb{Z}_{\mathcal{U}}$ is a constant system on \mathcal{K} . Let us fix a marking $\tau : R^2k_*\mathbb{Z}_{\mathcal{U}} \simeq \Lambda \times \mathcal{K}$, where $\Lambda = \Lambda_{K3}$, and consider as before the resulting period map

$$p : \mathcal{K} \rightarrow \mathcal{D} = \{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\} \subset \mathbb{P}(\Lambda \otimes \mathbb{C}) \simeq \mathbb{P}^{21}.$$

Then, p is a local isomorphism by the local Torelli Theorem and by the fact that $\dim \mathcal{K} = \dim \mathcal{D} (= 20)$. Therefore, we may identify \mathcal{K} with an open neighbourhood $\mathcal{U} \subset \mathcal{D}$ of $p(0)$ by p . Since our argument is completely local, by abuse of notation, we write this \mathcal{U} again by \mathcal{D} and identify therefore $\mathcal{K} = \mathcal{D}$ by p . Let us write a general fiber of $\varphi : J \rightarrow \mathbb{P}^1$ by E and choose an integral basis S_i ($i = 1, \dots, r$) of $MW^0(\varphi)$, where $r := r(\varphi)$. Then S_i and the zero section O are all non-singular rational curves and E is an elliptic curve such that $(S_i, E) = (O, E) = 1$. For this purpose, we may assume that $r \leq 17$. By the definition of the Mordell-Weil lattice $(MW(\varphi), \langle *, * \rangle)$, we have the (minus sign of) isometric injective homomorphism $\iota : MW^0(\varphi) \hookrightarrow NS(J)$ given by $S \mapsto S - O$ [Sh2]. Then, $\langle S_i, S_i \rangle = 4 + 2(S_i, O)$. Note also that E, O, S_i are linearly independent in $NS(J)$ [Sh1]. Let us consider $(r + 2)$ elements in Λ_{K3} given by $e := \tau([E])$, $o := \tau([O])$ and $s_i := \tau([S_i])$. Then, these are also linearly independent in Λ . Consider the subset \mathcal{L} of \mathcal{D} defined by $(e, *) = (o, *) = (s_i, *) = 0$. This is a smooth analytic subset of \mathcal{D} of dimension $20 - (r + 2)$ which contains $p(0)$.

This follows from the non-degeneracy of $(*,*)|\langle o, e, s_i \rangle^\perp$ and the Lefschetz $(1,1)$ -Theorem. Through the identification made above, we may regard $0 \in \mathcal{L} \subset \mathcal{K}$. Then we may speak of the family $\tilde{j} : \tilde{\mathcal{J}} \rightarrow \mathcal{L}$ obtained as the restriction of $k : \mathcal{U} \rightarrow \mathcal{K}$ to \mathcal{L} . Then by [Ko, Theorem 14] (see also [MM]) or by [Hu1, Section 1 (1.14)] in more sophisticated terminology, we see that \mathcal{L} is the locus on which the invertible sheaves $\mathcal{O}_J(E)$, $\mathcal{O}_J(O)$, $\mathcal{O}_J(S_i)$ on J lift to invertible sheaves \mathcal{E} , \mathcal{O} and \mathcal{S}_i on the whole space $\tilde{\mathcal{J}}$. Since $20 - (r + 2) \geq 1$ by $r \leq 17$, we can take a sufficiently small disk $0 \in \Delta \subset \mathcal{L}$. Then we may consider the induced family $j : \mathcal{J} \rightarrow \Delta$. We denote the restrictions of \mathcal{E} , \mathcal{O} and \mathcal{S}_i on \mathcal{J} by the same letter. We also shrink Δ freely whenever it is convenient. Note that $\chi(\mathcal{O}_J(S_i)) = 1$, $h^0(\mathcal{O}_J(S_i)) = 1$ and $h^q(\mathcal{O}_J(S_i)) = 0$ for $q > 0$, because S_i is a smooth rational curve on a K3 surface. Then by applying the upper semi-continuity of coherent sheaves and by the Theorem of cohomology, we see that $j_*\mathcal{S}_i$ are invertible sheaves which satisfy the base change property. Then $(j_*\mathcal{S}_i) \otimes \mathbb{C}(0) \simeq H^0(\mathcal{O}_J(S_i))$. Therefore, by Nakayama's Lemma, all of C_i lift not only as invertible sheaves but also as effective divisors on \mathcal{J} . By abuse of notation, we denote these divisors again by \mathcal{S}_i . Since the smoothness is an open condition for a proper morphism, $\pi|\mathcal{S}_i : \mathcal{S}_i \rightarrow \Delta$ is also smooth. Combining this with the fact that small deformation of \mathbb{P}^1 is again \mathbb{P}^1 , we see that $S_{i,t} := \mathcal{S}_i|_{\mathcal{J}_t}$ is again a smooth rational curve on \mathcal{J}_t for all $t \in \Delta$. The same holds for $\mathcal{O}_t := \mathcal{O}|_{\mathcal{J}_t}$. Note that $\chi(\mathcal{O}_J(E)) = 2$, $h^0(\mathcal{O}_J(E)) = 2$ and $h^q(\mathcal{O}_J(E)) = 0$ for $q > 0$, because E is an elliptic curve on a K3 surface. Then, $h^q(\mathcal{E}|_{\mathcal{J}_t}) = 0$ and $h^0(\mathcal{E}|_{\mathcal{J}_t}) = 2$. Therefore, $j_*\mathcal{E}$ is a locally free sheaf of rank 2 which satisfies the base change property. In particular, $j^*j_*\mathcal{E}|_J = H^0(\mathcal{O}_J(E))$. Since $\mathcal{O}_J(E)$ is globally generated, we see again by Nakayama's Lemma that the natural map $j^*j_*\mathcal{E} \rightarrow \mathcal{E}$ is also surjective. Therefore we may take a morphism $\Phi : \mathcal{J} \rightarrow \mathcal{W}$ over Δ associated to this surjection. Then, by the base change property, we find that the restriction $\Phi_t : \mathcal{J}_t \rightarrow \mathcal{W}_t$ coincides with the morphism given by the surjection $H^0(\mathcal{E}|_{\mathcal{J}_t}) \otimes \mathcal{O}_{\mathcal{J}_t} \rightarrow \mathcal{E}|_{\mathcal{J}_t}$. This is an elliptic fibration by $h^0(\mathcal{E}|_{\mathcal{J}_t}) = 2$ and by the adjunction formula on a K3 surface. Therefore, the factorisation $\Phi : \mathcal{J} \rightarrow \mathcal{W}$ makes $j : \mathcal{J} \rightarrow \Delta$ a family of elliptic surfaces over Δ . By the invariance of the intersection number, we have $(\mathcal{S}_{i,t}, \mathcal{E}_t) = (S_i, E) = 1$. Therefore, $S_{i,t}$ is also a section of Φ_t . The same holds for \mathcal{O}_t . Therefore $\Phi : \mathcal{J} \rightarrow \mathcal{W}$ makes $j : \mathcal{J} \rightarrow \Delta$ a family of Jacobian K3 surfaces with zero section \mathcal{O} . Moreover, by passing to the Weierstrass family over Δ given by \mathcal{O} and using the characterisation of $MW^0(\varphi)$ that $S \in MW(\varphi)$ is in $MW^0(\varphi)$ if and only if S does not meet the singular points of the Weierstrass model, we also see that $\mathcal{S}_{i,t}$ are all in $MW^0(\Phi_t)$. In addition, the intersection matrix of \mathcal{E}_t , \mathcal{O}_t , $\mathcal{S}_{i,t}$ are the same as the one for E , O , S_i in Λ and is then hyperbolic. Therefore, \mathcal{E}_t , \mathcal{O}_t , $\mathcal{S}_{i,t}$ are also linearly independent in $H^2(\mathcal{J}_t, \mathbb{Z})$. Hence so are in $NS(\mathcal{J}_t)$. Thus by the injection $MW^0(\Phi_t) \hookrightarrow NS(\mathcal{J}_t)$ quoted above, we see that $\mathcal{S}_{i,t}$ are also linearly independent in $MW^0(\Phi_t)$. In particular, $r(\Phi_t) \geq r$ for all $t \in \Delta$. Since the base space Δ is chosen in the Kuranishi space, our family $j : \mathcal{J} \rightarrow \Delta$ is not trivial. Therefore, by Corollary 6, there exists $t_0 \in \Delta$ such that $r(t)$ is strictly upper semi-continuous at t_0 . In particular, $r(t_0) > r$. By the invariance intersection and by the relation between $\langle *, * \rangle$ and $(*, *)$ quoted above, we see that the map $a : MW^0(\varphi) \rightarrow MW^0(\Phi_{t_0})$ given by $S_i \mapsto \mathcal{S}_{i,t_0}$ is then an isometric injection. If $r(t_0) = r + 1$, then we may define $\varphi_{r+1} : \mathcal{J}_{r+1} \rightarrow \mathbb{P}^1$ to be this Jacobian K3 surface $\Phi_{t_0} : \mathcal{J}_{t_0} \rightarrow \mathbb{P}^1$. Let us treat the case where $r(t_0) \geq r + 2$. Since $18 \geq r(t_0)$, we have $16 \geq r$. For simplicity, we abbreviate $\Phi_{t_0} : \mathcal{J}_{t_0} \rightarrow \mathbb{P}^1$ and $r(t_0)$ by $\varphi' : \mathcal{J}' \rightarrow \mathbb{P}^1$ and \mathcal{J}' respectively. We denote the image of the basis S_i ($1 \leq i \leq n$) of $MW^0(\varphi)$ in

$MW^0(\varphi')$ by the same letters S_i and take $T_j \in MW^0(\varphi')$ $j = r+1, r+2, \dots, r'$ such that S_i and T_j form a basis of $MW^0(\varphi') \otimes \mathbb{Q}$ over \mathbb{Q} . (Here note that our embedding might not be primitive so that we can not prolong S_i to an integral basis of $MW^0(\varphi')$ in general.) Let us consider the Kuranishi space \mathcal{K}' of J' and take the subspace $\mathcal{L}' \subset \mathcal{K}'$ defined by the fibre class E' of φ' , the zero section O , all of S_i , and T_{r+1} , and denote by $j' : \mathcal{J}' \rightarrow \mathcal{L}'$ the family induced by the Kuranishi family as before. Then, $\dim \mathcal{L}' = 20 - (2 + r + 1) > 0$, because E', O, S_i and T_{r+1} are linearly independent in $H^2(J', \mathbb{Z})$ and because $r \leq 16$. In addition, considering \mathcal{L}' as a subspace in the period domain under the identification made as before, and applying the same argument as in (1.1) based on the holomorphicity of the period mapping and the Lefschetz (1,1)-Theorem, we see that the Néron-Severi group of any general fibre \mathcal{J}'_t in the sense of the I -topology (Here we define a non-empty open set as a complement of countably many proper analytic subsets) is isomorphic to the primitive closure of $\mathbb{Z}\langle E_{18}, O, S_i, T_{r+1} \rangle$ in $H^2(J', \mathbb{Z})$. In particular, $\rho(\mathcal{J}'_t) = r + 3$ for general t . Moreover, by the same argument as above, we find that this family becomes a family of Jacobian K3 surfaces $\mathcal{J}' \xrightarrow{\Phi'} \mathcal{W}' \rightarrow \mathcal{L}'$ such that each fibre $\Phi'_t : \mathcal{J}'_t \rightarrow \mathcal{W}'_t$ satisfies that $MW^0(\varphi) \subset MW^0(\Phi'_t)$ and that $r(\Phi'_t) \geq r + 1$. On the other hand, we have $r(\Phi'_t) \leq r + 1$ for general t by Shioda's formula and by $\rho(\mathcal{J}'_t) = r + 3$. Then we have $r(\Phi'_t) = r + 1$ and may define $\varphi_{r+1} : J_{r+1} \rightarrow \mathbb{P}^1$ to be $\Phi'_t : \mathcal{J}'_t \rightarrow \mathcal{W}'_t$ for general t . The first statement of Corollary 8 now follows from induction on $(18 - r)$. Next we shall show the middle statement of Corollary 8. Let $\phi' : S' \rightarrow \mathbb{P}^1$ be a Jacobian K3 surface such that $\text{rank}(\phi') = 18$ and M a sublattice of $MW(\phi')$. Then, by taking a general point of the locus of the Kuranishi space defined by the basis of M , zero section of ϕ' and general fibre of ϕ' as before, we obtain a Jacobian K3 surface $\phi : S \rightarrow \mathbb{P}^1$ such that $M \subset MW^0(\phi)$ and $r(\phi) = r$. Finally, we show the last assertion of Corollary 8. Assume that $M \subset MW^0(\phi)$ and is of finite index. Since the pairing $\langle *, * \rangle$ is integral valued on $MW^0(\phi)$ [Sh2], we have then $M \subset MW^0(\phi) \subset M^*$. Since $M \subset M^*$ is of finite index, the possibilities of $MW^0(\phi)$ is then only finitely many. By [Sh2], we have also $MW^0(\phi) \subset MW(\phi)/(\text{torsion}) \subset MW^0(\phi)^*$. Therefore each given $MW^0(\phi)$ also recovers $MW(\phi)/(\text{torsion})$ up to finitely many ambiguities. Now it is sufficient to show the boundedness of the torsion subgroups of Jacobian K3 surfaces. For those which have non-constant j -invariant, the result follows from the classification due to Cox [Co]. Let us consider the case where the j -invariant is constant. Note that a Jacobian K3 surface always admits at least one singular fibre, because its topological Euler number is positive. Therefore, by the classification of the singular fibres whose j -values are not ∞ and by the general fact that the specialisation map $MW(\phi)_{\text{torsion}} \rightarrow (\phi^{-1}(t))_{\text{reg}}$ is injective, we see that the possible torsion groups are: $0, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4$ or $(\mathbb{Z}/2)^{\oplus 2}$. This implies the result. \square

§3. CONSTRUCTION OF EXAMPLES

We shall give explicit construction of Examples in Introduction.

Construction of Example 4. By [OZ] based on [SI], there exists a K3 surface S of $\rho(S) = 20$ which contains 19 smooth rational curves C_i ($i = 1, 2, \dots, 19$) whose configuration is of Type D_{19} and is primitive in $NS(S)$. Consider the Kuranishi space of S and take the locus defined by $(C_i \cdot *) = 0$ ($1 \leq i \leq 19$) as in the proof of Corollary 8. Then this is a smooth curve and we may take a small curve Λ inside

this curve and speak of the family $f : \mathcal{X} \rightarrow \Delta$ induced by the Kuranishi family. Then, by the same argument as in Corollary 8, we see that $\rho(t) \geq 19$ for $t \in \Delta$ and that if $\rho(t) = 19$, then $NS(\mathcal{X}_t) \simeq D_{19}$. In the last case, we have $a(\mathcal{X}_t) = 0$, because D_{19} is negative definite. Note that f is not trivial. Then by Theorem 1, we have the dense countable set \mathcal{S} such that $\rho(\mathcal{X}_s) = 20$ for $s \in \mathcal{S}$ and $\rho(\mathcal{X}_t) = 19$ for $t \notin \mathcal{S}$. Since a K3 surface of the maximal Picard number 20 is algebraic, this family satisfies all the properties required. \square

Construction of Example 5. Applying the construction in Corollary 3 for $\varphi(z) \equiv \sqrt{-1}$, we obtain a family of Kummer surfaces $f : \mathcal{X} \rightarrow \Delta$ such that $\mathcal{X}_z = \text{Km}(E_z \times E_{\sqrt{-1}})$. Then by [SM], we have $\rho(\mathcal{X}_z) = 18$ for $z \notin \mathbb{Q}(\sqrt{-1})$, and $\rho(\mathcal{X}_z) = 20$ for $z \in \mathbb{Q}(\sqrt{-1})$. Since $\mathbb{Q}(\sqrt{-1})$ is dense in Δ , this provides an example we seeked. \square

Construction of Example 7. Let us first consider the family of rational Jacobian surfaces with rational double points $h : \mathcal{Z} \rightarrow \mathbb{P}^1 \times \Delta_u \rightarrow \Delta_u$ defined by the Weierstrass equation $y^2 = x^3 + ux + s^5$. Here u is the coordinate of Δ and s is the inhomogeneous coordinate of \mathbb{P}^1 . Then either by the Néron algorithm or by a direct calculation, we can easily check the following fact: \mathcal{Z} is smooth; \mathcal{Z}_u ($u \neq 0$) is smooth and $\mathcal{Z}_u \rightarrow \mathbb{P}^1$ has singular fibres of Type I_1 over $4u^3 + 27s^{10} = 0$ and of Type II over $s = \infty$; and $\mathcal{Z}_0 \rightarrow \mathbb{P}^1$ has one singular point of type E_8 over $s = 0$ and has a singular fibre of Type II over $s = \infty$. Therefore taking an appropriate finite covering $\Delta_v \rightarrow \Delta_u$ and a simultaneous resolution of the pull back family, we obtain a family of smooth rational Jacobian surfaces $g : \mathcal{Y} \rightarrow \mathbb{P}^1 \times \Delta_v \rightarrow \Delta_v$ such that $\mathcal{Y}_v \rightarrow \mathbb{P}^1$ ($v \neq 0$) has 10 singular fibre of Type I_1 and one singular fibre of Type II , and $\mathcal{Y}_0 \rightarrow \mathbb{P}^1$ has one singular fibre of Type II^* and one singular fibre of Type II . Then by Shioda's formula, we have $r(v) = 8$ for $v \neq 0$ and $r(0) = 0$. Let us choose large number M such that the divisor $s = M$ on $\mathbb{P}^1 \times \Delta_v$ does not meet the discriminant locus. This is possible by the description above. Let us take the double covering $\mathbb{P}^1 \times \Delta_t \rightarrow \mathbb{P}^1 \times \Delta_v$ ramified over $s = M$ and $s = \infty$ and consider the relatively minimal model $f : \mathcal{X} \rightarrow \mathbb{P}^1 \times \Delta_t \rightarrow \Delta_t$ of the pull back family $f' : \mathcal{X}' \rightarrow \mathbb{P}^1 \times \Delta_t \rightarrow \Delta_t$ over Δ_t . Note that \mathcal{X}' is equi-singular along the preimage of the cuspidal points of fibres $(\mathcal{Y}_v)_\infty$ of $\mathcal{Y}_v \rightarrow \mathbb{P}^1$. Then, by the monodromy calculation, we see that $f : \mathcal{X} \rightarrow \mathbb{P}^1 \times \Delta_t \rightarrow \Delta_t$ is a smooth family of Jacobian K3 surfaces such that $\mathcal{X}_0 \rightarrow \mathbb{P}^1$ has two singular fibres of Type II^* and one singular fibre of Type IV ; and $\mathcal{X}_t \rightarrow \mathbb{P}^1$ ($t \neq 0$) has 20 singular fibres of Type I_1 and one singular fibre of Type IV . Note also that $r(t) \geq r(v) = 8$ for $t \neq 0$. On the other hand, again by Shioda's formula, we have $20 \geq \rho(t = 0) = 2 + r(t = 0) + (9 - 1) + (9 - 1) + (3 - 1)$. Therefore $\rho(0) = 20$ and $r(0) = 0$ for the centre fibre of f . Moreover, this family $\mathcal{X} \rightarrow \Delta$ is not trivial even as a family of K3 surfaces. Indeed, otherwise, we have $\mathcal{X} \simeq \mathcal{X}_0 \times \Delta_t$. Since $\text{Pic}(\mathcal{X}_0)$ is discrete, it is impossible for \mathcal{X}_0 to admit a family of elliptic fibrations which vary continuously. Then, our family $\mathcal{X} \rightarrow \Delta$ must be also trivial as a family of elliptic fibre spaces if $\mathcal{X} \simeq \mathcal{X}_0 \times \Delta_t$. However, this contradicts the fact that the type of singular fibre of \mathcal{X}_0 and \mathcal{X}_t are different. Therefore our family is not trivial even as a family of K3 surfaces. Hence, by (1.2), we have $\rho(t) < 20$ for general t in the I -topology. \square

Correction of [OS]. Taking this place, we shall quote a correction of mistakes of [OS Main Theorem] pointed out by Y. Yamada:

No. 12 : the first line of the matrix of $E(K)^0$ should be $4, -1, 0, 1$

No 32 : $E(K)^0$ should be $\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$.

No 70 : $E(K)$ should be $\mathbb{Z}/4\mathbb{Z}$. \square

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